The Man Who Knew Infinity

Edmund Y. M. Chiang

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Capstone Project course

Department of Mathematics

香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

Outline

Bertrand Russell

- "Mathematics, rightly viewed, possesses not only truth, but supreme beauty–a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show."

Bertrand Russell, A History of Western Philosophy

??

- "An equation means nothing to me unless it expresses a thought of god".

A genius ?

- Ramanujan was born on 22$^{\text{nd}}$ December 1887 in *Erode*, a small town about 250 miles southwest of Chennai (Madras), India. He died on 26$^{\text{th}}$ April 1919 when he was 32 years old.
- He wrote some of the most famous letters in the history of mathematics. He wrote down thousands of identities.
- He is generally acknowledged as one of the greatest Indian mathematicians throughout the history.
- He is ranked along with L. Euler and K. Jacobi. Some even said Gauss.
- "... the most romantic figure in the recent history of mathematics ..."
- He worked in Number theory, Analysis (partition functions, elliptic functions, continued fraction, etc) yet he only received little formal education.

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Zurich Film Festival (2015)

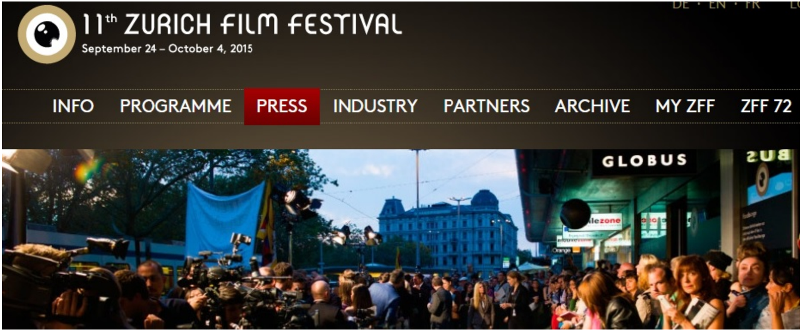


Figure: Zurich Film Festival Opening Film

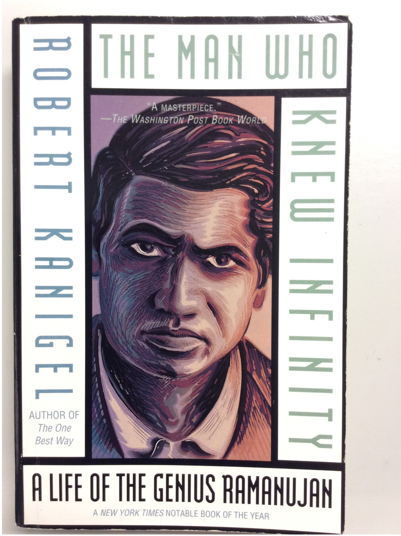
The Man Who Knew Infinity film (2015)

FILM 08.25.15 | 06:43AM PT

'The Man Who Knew Infinity' to Open Zurich Film Festival

BY LEO BARRACLOUGH

Robert Kanigel's book (1991)



Early Life (I)

- 1 year old: moved to a Hindu town Kumbakonam (Pop: 50,000) 170 miles south of Chennai, where his father was a cloth merchant's clerk. Entered school at five years old.
- 11 years old: Ramanujan did well in all school subjects in a town high school, especially in mathematics.
- 13 years old: Ramanujan started to research on geometric series. He had tried to solve quartic equation.
- Things take on a dramatic turn after a friend lent Ramanujan a Government College library's copy of G. S. Carr's "Synopsis of Elementary Results in Pure Mathematics"
- Hardy's comment:

> *"... It contains enunciation of 6,165 theorems, systematically and quite scientifically arranged, with proofs which are often little more than cross-references ..."*

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Early Life (II)

- Ramanujan became addicted to mathematics and start recording his own results in Carr's format.
- By 1904, i.e., 17 years old, Ramanujan had begun investigated the series $\sum \frac{1}{n}$ and calculated Euler's constant to 15 decimal places.
- The same year his school discontinued his scholarship for poor performance of other subjects.
- In 1906, he was given a second scholarship to attend Pachaiyappa's College in Madras in preparing for entering the University of Madras. But he failed all subjects except mathematics.

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G. H. Hardy

Figure: 1877-1947

A letter from an Indian clerk

In 1913 (16th January) Ramanujan wrote to G. H. Hardy. He introduced himself and his work as:

"Dear Sir, I beg to introduce myself to you as a clerk in the Accounts Department ... on a salary of only \$20 per annum ... I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at mathematics.I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'."

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Hardy's reaction

- "His letters contained the bare statement of about 120 theorems. Several of them were known already, others were not. Of these, some I could prove (after harder work than I had expected) while others fairly blew me away. I had never seen the like! Only a mathematician of the highest class could have written them. They had to be true, for if they were not, no one would have the imagination to invent them. A few were definitely wrong. But that only added credence to my feeling that the writer was totally honest, since great mathematicians are commoner than frauds of the incredible skill that would be needed to create such a letter. "
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About 120 formulae I

(1)
$$\frac{1}{1^3}\cdot\frac{1}{2^1}+\frac{1}{2^3}\cdot\frac{1}{2^2}+\frac{1}{3^3}\cdot\frac{1}{2^3}+\frac{1}{4^3}\cdot\frac{1}{2^4}+\cdots$$
$$=\frac{1}{6}(\log 2)^3-\frac{\pi^2}{12}\log 2+\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right).$$

(2)
$$1+9\cdot\left(\frac{1}{4}\right)^4+17\cdot\left(\frac{1\cdot 5}{4\cdot 8}\right)^4+25\cdot\left(\frac{1\cdot 5\cdot 9}{4\cdot 8\cdot 12}\right)^4+\cdots=\frac{2\sqrt{2}}{\sqrt{\pi}\{\Gamma(\frac{3}{4})\}^2}.$$

(3)
$$1-5\cdot\left(\frac{1}{2}\right)^3+9\cdot\left(\frac{1\cdot 3}{2\cdot 4}\right)^3-\cdots=\frac{2}{\pi}.$$

(4)
$$\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\cdots=\frac{1}{24}.$$

(5)
$$\frac{\coth\pi}{1^7}+\frac{\coth 2\pi}{2^7}+\frac{\coth 3\pi}{3^7}+\cdots=\frac{19\pi^7}{56700}.$$

(6)
$$\frac{1}{1^5\cosh\frac{\pi}{2}}-\frac{1}{3^5\cosh\frac{3\pi}{2}}+\frac{1}{5^5\cosh\frac{5\pi}{2}}-\cdots=\frac{\pi^5}{768}.$$

(7)
$$\frac{1}{(1^2+2^2)(\sinh 3\pi-\sinh\pi)}+\frac{1}{(2^2+3^2)(\sinh 5\pi-\sinh\pi)}$$
$$+\frac{1}{(3^2+4^2)(\sinh 7\pi-\sinh\pi)}+\cdots=\frac{1}{2\sinh\pi}\left(\frac{1}{\pi}+\coth\pi-\frac{\pi}{2}\tanh^2\frac{\pi}{2}\right).$$

Figure: 1913 (B. Berndt)

About 120 formulae II

(5) If $\alpha\beta = \pi$, then

$$\sqrt{\alpha}\int_0^\infty \frac{e^{-x^2}}{\cosh \alpha x}dx = \sqrt{\beta}\int_0^\infty \frac{e^{-x^2}}{\cosh \beta x}dx.$$

(6) If $\alpha\beta = \pi^2$, then

$$\frac{1}{\sqrt[4]{\alpha}}\left\{1 + 4\alpha\int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}dx\right\} = \frac{1}{\sqrt[4]{\beta}}\left\{1 + 4\beta\int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x}-1}dx\right\}.$$

(7)

$$n\left(e^{-n^2} - \frac{e^{-\frac{1}{3}n^2}}{3\sqrt{3}} + \frac{e^{-\frac{1}{5}n^2}}{5\sqrt{5}} - \cdots\right)$$
$$= \sqrt{\pi}(e^{-n\sqrt{\pi}}\sin n\sqrt{\pi} - e^{-n\sqrt{3\pi}}\sin n\sqrt{3\pi} + \cdots).$$

(8) If n is any positive integer excluding 0,

$$\frac{1^{4n}}{(e^\pi - e^{-\pi})^2} + \frac{2^{4n}}{(e^{2\pi} - e^{-2\pi})^2} + \frac{3^{4n}}{(e^{3\pi} - e^{-3\pi})^2} + \cdots$$
$$= \frac{n}{\pi}\left\{\frac{B_{4n}}{8n} + \frac{1^{4n-1}}{e^{2\pi}-1} + \frac{2^{4n-1}}{e^{4\pi}-1} + \frac{3^{4n-1}}{e^{6\pi}-1} + \cdots\right\}$$

where $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $\ldots$.

(7)

Figure: 1913 (B. Berndt)

About 120 formulae III

$$-\ \frac{\ }{1000}$$

(6) $$\int_0^a e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} - \frac{e^{-a^2}}{2a\,+}\ \frac{1}{a\,+}\ \frac{2}{2a\,+}\ \frac{3}{a\,+}\ \frac{4}{2a\,+}\ \cdots$$

(7) The coefficient of x^n in

$$\frac{1}{1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \cdots}$$

$$= \text{ the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n}) - \frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}.$$

(9)

IX. Theorems on continued fractions, a few examples are:—

(1) $$\frac{4}{x\,+}\ \frac{1^2}{2x\,+}\ \frac{3^2}{2x\,+}\ \frac{5^2}{2x\,+}\ \frac{7^2}{2x\,+}\ \cdots = \left\{\frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+3}{4}\right)}\right\}^2.$$

(2) If

Figure: 1913 (B. Berndt)

Ramanujan was sick

- During his five-year stay in Cambridge, which unfortunately overlapped with the first World War years, he published 21 papers, five of which were in collaboration with Hardy.
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Fermat's $q-$integral (I)

- Before Leibniz and Newton, that is, before you have the Fundamental Theorem of Calculus,

$$\int_0^a x^\alpha \, dx = F(a) - F(0) = \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_0^a = \frac{a^{\alpha+1}}{\alpha+1}$$

  that is one can find a primitive of x^α.
- How did people compute $\int_0^a x^\alpha \, dx$ where α is rational?
- Fermat computed this around 1650s: Divide $[0, a]$ into subintervals of geometric dissection $\{x_n = aq^n\}_0^\infty$ so that ...

Fermat's q−integral (II)

$$\begin{aligned}\sum_{n=0}^{\infty} x_n^{\alpha}(x_n - x_{n+1}) &= \sum_{n=0}^{\infty}(aq^n)^{\alpha}(aq^n - aq^{n+1}) \\ &= a^{\alpha+1}(1-q)\sum_{n=0}^{\infty} q^{(\alpha+1)n} \\ &= \frac{a^{\alpha+1}(1-q)}{1-q^{\alpha+1}}.\end{aligned}$$

Writing $\alpha = \ell/m$ and $t = q^{1/m}$ shows the above is equal to

$$\begin{aligned}\frac{\alpha^{(\ell+m)/m}(1-t^m)}{1-t^{m+n}} &= \alpha^{(\ell+m)/m}\frac{1+t+\cdots+t^{m-1}}{1+t+\cdots+t^{m+\ell-1}} \\ &\to \left(\frac{m}{m+\ell}\right)\alpha^{(\ell+m)/m}, \quad \text{as } t \to 1 \ (q \to 1).\end{aligned}$$

Fermat's $q-$integral (III)

- The above suggests that

$$m \Longleftrightarrow 1+q+q^2+\cdots q^{m-1}=\frac{1-q^m}{1-q},$$

- so that

$$k!_q=\left(\frac{1-q}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\cdots\left(\frac{1-q^k}{1-q}\right)$$

- We rewrite

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{n!_q}{k!_q(n-k)!_q}$$

as the $q-$binomial coefficient.

- L'Hôspital's rule yields

$$\lim_{q^a\to 1}\frac{(q^a;\,q)_n}{(1-q)^n}=a(a+1)\cdots(a+n-1).$$

Set $a=1$ yields $n!$. Thus $n!_q$ is called $q-n$ factorial.

Binomial Identity (I)

- $$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

- $$(a+b)^3 = (a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

- $$\begin{aligned}(a+b)^4 &= (a+b)(a+b)(a+b)(a+b)\\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

- $$\begin{aligned}(a+b)^5 &= (a+b)(a+b)(a+b)(a+b)(a+b)\\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

Binomial Identity (II)

- $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

- $$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad k! = k \times (k-1) \times \cdots \times 2 \times 1$$

- $$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \times 2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = 10$$

- Combinatorial interpretation: The total number of ways to choose 3 objects out of 5 objects.
- So $\binom{n}{k}$ counts the total number of ways to choose k objects out of n objects.

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- $$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3! \times 2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = 10$$

- Combinatorial interpretation: The total number of ways to choose 3 objects out of 5 objects.
- So $\binom{n}{k}$ counts the total number of ways to choose k objects out of n objects.

Some formulae I

- Let $0 < q < 1$. We define
-
$$(a;\, q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k), \quad n \geq$$

-
$$(a;\, q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots = \prod_{k=0}^{\infty}(1-aq^k), \quad n \geq 1$$

- The infinite product always converge, a necessary condition is
$$(1-aq^{n-1}) \to 1, \quad as\ n \to \infty$$

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Some formulae II

- $$n \Longleftrightarrow 1+q+q^2+\cdots q^{n-1} = \frac{1-q^n}{1-q},$$

- $$k!_q = \left(\frac{1-q}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\cdots\left(\frac{1-q^k}{1-q}\right) := \frac{(q,;\,q)_k}{(1-q)^k}$$

- $q-$binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q(n-k)!_q}. \qquad (a;\,q)_n = (1-a)(1-aq)\cdots(1-aq^{n-k+1})$$

- If $|x|<1$, then

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$$

Rogers-Ramanujan Identities (1894, 1917, 1919)

Let $|q| < 1$:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;\, q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;\, q)_n} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})}$$

where $0 < q < 1$ and

$$(q;\, q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q;\, q)_0 = 1.$$

Binomial identity

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

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Verification of the first identity (I)

$$\prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)} \cdots$$
$$= (1+q+q^2+q^3+q^4+q^5+q^6+\cdots)$$
$$\times (1+q^4+q^8+q^{12}+\cdots)$$
$$\times (1+q^6+q^{12}+\cdots)$$
$$\times \cdots$$
$$= 1+q+q^2+q^3+2q^4+2q^5+3q^6+\cdots.$$

Verification of the first identity (II)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;\, q)_n} = \frac{1}{(q;\, q)_0} + \frac{q}{(q;\, q)_1} + \frac{q^4}{(q;\, q)_2} + \frac{q^9}{(q;\, q)_3} + \cdots$$

$$= 1 + \frac{q}{(1-q)} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \cdots$$

$$+ q(1 + q + q^2 + q^3 + q^4 + q^5 + q^6 + \cdots)$$

$$+ q^4(1 + q + q^2 + q^3 + \cdots)(1 + q^2 + q^4 + \cdots)$$

$$+ q^9(1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots)(1 + q^3 + q^6 + \cdots)$$

$$+ \cdots$$

$$= 1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + \cdots .$$

Interpretations of the Identities (I)

Theorem

The right-hand side of the first Rogers-Ramanujan identity gives the partitions with parts congruent to $1 \pmod 5$ or congruent to $4 \pmod 5$.

Theorem

The left-hand side of the first Rogers-Ramanujan identity gives the partitions for which the difference between any two parts is at least two.

Corollary

The number of partitions of an integer N into parts in which the difference between any two parts is at least 2 is same as the number of partitions of N into parts congruent to 1 or $4 \pmod 5$.

Interpretations of the Identities (II)

Let us choose $N = 6$. The partitions in which the number of parts are congruent to 1 or $4 \pmod 5$ are

$$(1, 1, 1, 1, 1, 1), \ (4, 1, 1), \ (6).$$

The number of partitions in which the difference of whose parts is at least two are:

$$(4, 2), \ (5, 1), \ (6).$$

So both has three partitions.

Interpretations of the Identities (III)

Let us choose $N = 9$. The partitions in which the parts are congruent to 1 or 4 (mod) 5 are

$$(1, 1, 1, 1, 1, 1, 1, 1, 1),\ (4, 1, 1, 1, 1, 1)$$

$$(4, 4, 1),\ (6, 1, 1, 1),\ (9).$$

Then the number of partitions in which the difference of whose parts is at least two are:

$$(5, 3, 1),\ (6, 3),\ (7, 2),\ (8, 1),\ (9).$$

Both have five members.

History of the Rogers-Ramanujan identities

- Ramanujan conjectured such identities (1913 ?)
- L. J. Rogers had already proved them in 1894
- Rediscovered by the physicist R. J. Baxter in 1985 for his work of hard hexagon model in statistical mechanics.
- There are now many such identities found, but the understanding of them is still poor.

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Srinivasa Ramanujan (1887-1919)

Figure: He is regarded as a hero in India

What is a partition?

Let N be a positive integer. Then we define a partition of N to be

$$N = s_1 + s_2 + \cdots + s_k$$

so that $N \geq s_1 \geq s_2 \geq \cdots s_k$. That is, the s_j are just integers smaller or equal to N. For example, let us consider $N = 4$. Then

$$\begin{aligned} 4 &= 1 + 1 + 1 + 1 \\ &= 2 + 1 + 1 \\ &= 2 + 2 \\ &= 3 + 1 \\ &= 4. \end{aligned}$$

Hence there are 5 different partitions of 4. The are denoted as vectors (1, 1, 1, 1), (2, 1, 1), (2, 2), (3, 1) and finally (4). The numbers in the vectors above are called the parts of the partitions.

Examples: $N = 1$ to 8

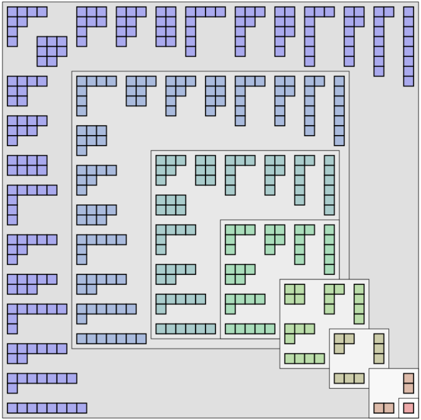

Figure: Wikipedia

Theorem

The number of partitions of integer n with at most m parts equals partitions of n in which no part exceeds m.

How big can $p(n)$ be?

n	a(n)
0	1
1	1
2	2
3	3
4	5
5	7
6	11
7	15
8	22
9	30
10	42
11	56
12	77
13	101
14	135
15	176
16	231
17	297
18	385
19	490
20	627
21	792
22	1002
23	1255
24	1575
25	1958
26	2436
27	3010
28	3718
29	4565
30	5604
31	6842
32	8349
33	10143

Figure: On-Line Encyclopedia of Integer Sequences (OEIS)

$p(n)$ can be very large

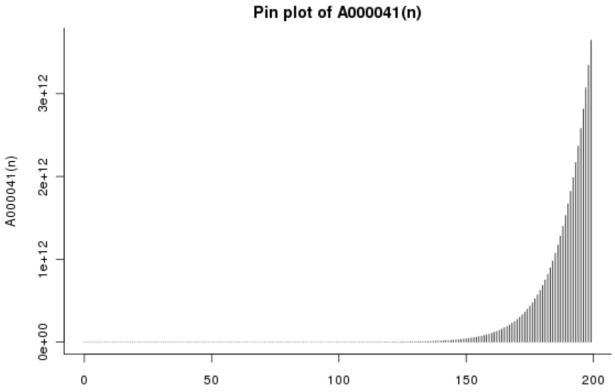

Figure: On-Line Encyclopedia of Integer Sequences (OEIS)

- $p(100) = 190,569,292$
- $p(200) = 3,972,999,029,388$ (MacMahon)
- $p(1000) = 24,061,467,864,032,622,473,692,149,727,991$

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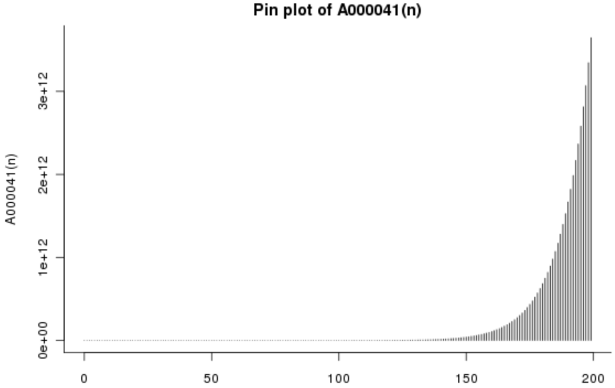

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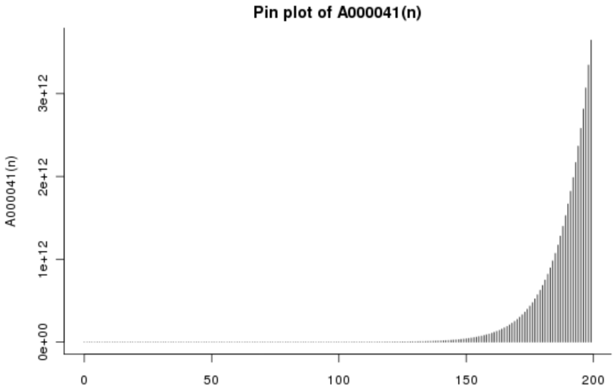

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Generating function

Recall that when $|q| < 1$

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + \cdots\,, \quad \frac{1}{1-q^2} = 1 + q^2 + q^4 + \cdots\,,$$
$$\frac{1}{1-q^3} = 1 + q^3 + q^6 + \cdots\,, \quad \ldots$$

Observe that (assuming convergence)

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = (1 + q + q^2 + q^3 + \cdots) \times (1 + q^2 + q^4 + \cdots) \times (1 + q^3 + q^6 + \cdots) \times \cdots$$

$$= 1 + p(1)q + p(2)q^2 + p(3)q^3 + \cdots + p(n)q^n + \cdots$$

That is, the coefficient $p(n)$ is precisely the partition function for integer n. This is due to Euler.

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Generating function (cont.)

- We note: The 1st factor counts "1" in the partition, the second factor counts "2" in the partition, etc. So every partition of n contributes exactly 1 to the coefficient of q^n.
- Hardy-Ramanujan used the idea of generating function to obtain an approximate formula when n is large (asymptotic)

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right), \quad n \to \infty.$$

- They later obtained a better approximation that it is so good that when $n = 200$, the approximation is only within the error 0.004 from the answer for $p(200)$ given by earlier by MacMahon. But then the approximation becomes exact in this case.

MacMahon's table

- Observe

1	1	2	3	5
7	11	15	22	30
42	56	77	101	135
176	231	297	385	490
627	792	1002	1255	1575
1958	2436	3010	3718	4565
5604	6842	8349	10143	12310

- We observe that the numbers in the last column are all multiple of 5.

Ramanujan's congruence results

Ramanujan stated

> *I have proved a number of arithmetic properties of $p(n)$... in particular, that*
>
> $$p(5n+4) \equiv 0 \mod 5,$$
> $$p(7n+5) \equiv 0 \mod 7,$$
>
> *... I have since found another method which enables me to prove all of these properties and a variety of others, of which the most striking is*
>
> $$p(11n+6) \equiv 0 \mod 11, \ldots$$

Atkin found $p(11^3 \cdot 13n + 237) \equiv 0 \mod 13$. If $\delta = 5^a 7^b 11^c$, $24\lambda \equiv 1 \mod \delta$, then $p(\delta n + \lambda) \equiv 0 \mod \delta$.

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After Ramanujan's death

- Hardy compiled Ramanujan's research papers and published a book entitled "Ramanujan: Twelve lectures on subjects suggested his life and work"
- In the beginning, Hardy wrote: "I have set myself a task in these lectures which is genuinely difficult and which, if I were determined to begin by making every excuse for failure, ... and try to help you to from, some sort of reasoned estimate of the most romantic figure in the recent history of mathematics; a man whose career seems full of paradoxes and contradictions, who defies almost all the canons by which we are accustomed to judge one another, and about whom all of us will probably agree in one judgment only, that he was in some sense a very great mathematician."

Srinivasa Ramanujan (1887-1919)

Figure: He is regarded as a hero in India

Berndt: Ramanujan's Notebooks

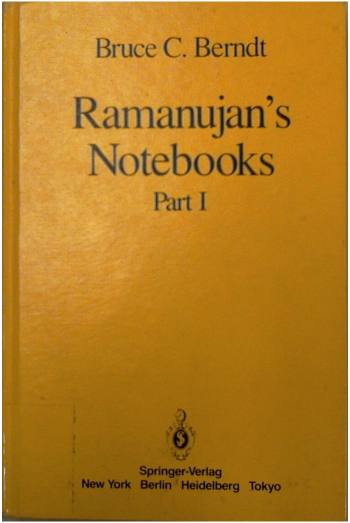


Figure: Five volumes were published

Bruce Berndt

Ramanujan's Lost Notebooks

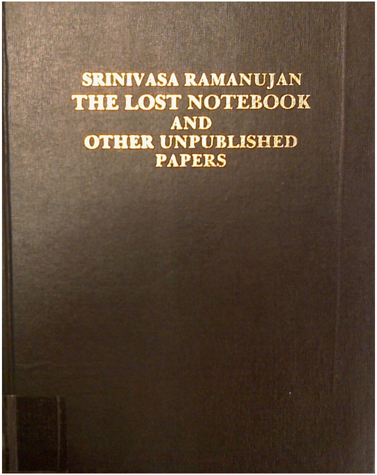
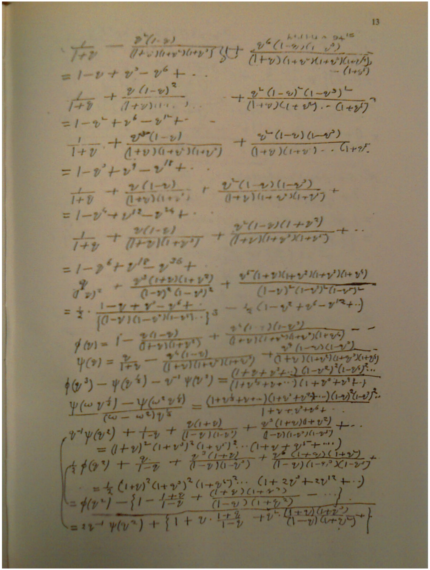


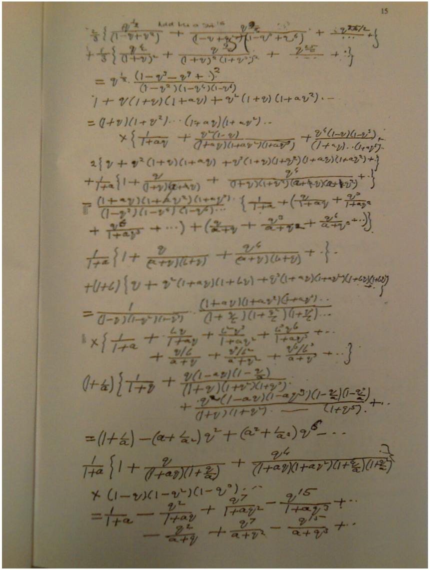
Figure: Andrew discovered in 1976 in Cambridge

G. Andrews

Andrew's Ramanujan Lost e.g. page 1



Andrew's Ramanujan Lost e.g. page 2



Andrews + Berndt: Ramanujan's Notebooks

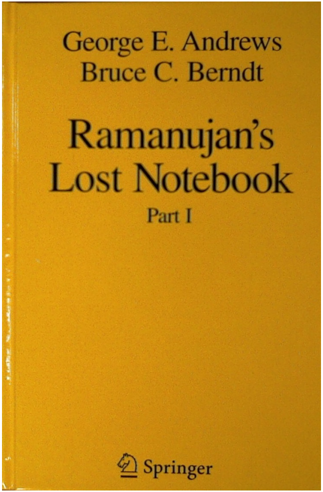
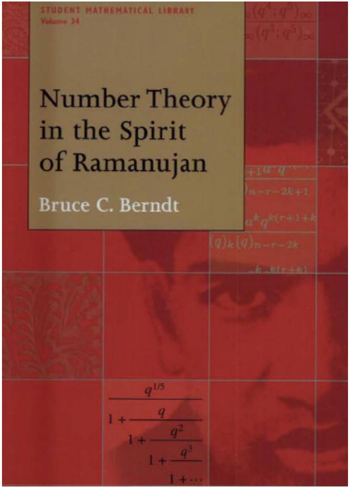


Figure: Four volumes have been published

An AMS book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Volume 34
Number Theory in the Spirit of Ramanujan
Bruce C. Berndt

Exercises

1 Prove that if a_n are real, and $\sum a_n$ is convergent, the product $\prod(1+a_n)$ converges, or diverges to zero, according as $\sum a_n^2$ converges or diverges.

2 Use the method of obtaining Stirling's formula to show that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}=2\sqrt{n}+C+\frac{1}{2\sqrt{2}}+O\Big(\frac{1}{n^{3/2}}\Big),$$

where

$$C=-(1+\sqrt{2})\Big(1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots\Big).$$

Thank you